Dirac theory in spacetime algebra: I. The generalized bivector Dirac equation

2001 J. Phys. A: Math. Gen. 34 1991

(http://iopscience.iop.org/0305-4470/34/10/304)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 1991–2005 www.iop.org/Journals/ja PII: S0305-4470(01)13285-3

Dirac theory in spacetime algebra: I. The generalized bivector Dirac equation

William P Joyce

Department of Physics and Astronomy, University of Canterbury, Private Bag 4800, Christchurch, New Zealand

Received 13 April 2000

Abstract

This paper formulates the standard Dirac theory without resorting to spinor fields. Spinor fields mix bivectors and vectors which have different properties in spacetime algebra. Instead the Dirac field is formulated as a generalized bivector field. All the usual results of the standard Dirac theory fall out naturally and simply. The plane-wave solutions to the Dirac equation are given and found to give eight independent solutions. The solutions correspond to particle/anti-particle energy states, spin $\pm\frac{1}{2}$ along the direction of propagation and two degrees of transverse polarization. Each solution has a double degeneracy corresponding to an internal $U(1) \otimes SU(2)$ symmetry of the Dirac equation. One further advantage of this formalism is that it is completely formulated without using a matrix representation of Clifford algebra instead of utilizing the inherent geometric meaning of the algebra.

PACS numbers: 0350D, 0330

1. Introduction

The history of Clifford algebra is a long one, originating from the work of Grassmann [10] on exterior algebra in the 1840s. However, it was not until 1878 that Clifford [2] defined these algebras. It was the Dirac equation that ultimately elevated Clifford algebra to prominence. More recently through the work of Hestenes [4–6], Clifford algebra has been shown to provide the framework for a geometric algebra. This algebra Hestenes calls spacetime algebra. This paper continues this trend by formulating the Dirac equation as a generalized bivector theory in spacetime algebra, thus eliminating the need for spinors. Spinors are undesirable in a geometric algebra being ideals in the algebra that mix vectors and bivectors. It should be noted that a real Dirac theory does not allow one to formulate the Dirac equation as a generalized bivector theory.

The paper begins by reviewing the two Clifford algebras capable of representing the geometric algebra corresponding to Minkowski spacetime and the properties of these Clifford algebras required for this paper. To take account of the Lorentz signature a general factor η is introduced to account for the over all sign difference between the two choices. This allows one to compare the two choices at any stage of the development. The beauty of spacetime algebra is that the Clifford multivectors have geometric meanings that exist without reference to any matrix representation of the algebra. Moreover, there is nothing to be gained by introducing matrix representations when there is no mixing of vectors and bivectors.

Next we review the introduction of an electromagnetic potential and formulate Maxwell equations as a single compact equation. There is a large literature on this subject [7, 8].

Having dispensed with the preliminaries, the Dirac algebra is defined as the complexified space of generalized bivectors. This space forms a complex eight-dimensional Clifford algebra. Next we construct four (equivalent) forms of the Dirac equation on our Dirac algebra according to clearly stated assumptions. These assumptions require that the Klein–Gordon equation is satisfied and no special significance is given to any of the three spatial directions. The Dirac–Hestenes equation [6] includes the Clifford element $\gamma^1\gamma^2$ and makes no direct reference to γ^3. Thus giving special significance to the γ^3 direction.

Once a choice for the Dirac equation is made, the remainder of the paper constructs the Dirac probability current, considers its transformation properties including *CPT* and derives and interprets the plane-wave solution. In particular, the infinitesimal generators for boosts and rotations are determined. The transformation under parity is used to introduce chiral fields which leads to a grand Dirac equation invariant under $Pin(3)$.

The plane-wave solutions give eight independent solutions. The usual particle/anti-particle eigenstates of time reversal, spin $\pm\frac{1}{2}$ in the direction of propagation and two degrees of transverse polarization account for four of the eight independent solutions leaving a degeneracy of two. This degeneracy is a result of a global gauge invariance under $U(1)\otimes SU(2)$. The next paper in the series explores the implications for local gauge theory focusing on the $U(1)\otimes SU(2)$ gauge group and electroweak theory.

The multivector fields have spin properties that depend on how they transform under rotation. Multivector fields that transform through one-sided rotation transformations are spin-$\frac{1}{2}$ fields and through double-sided rotation transformations are integer spin fields. It is important to note that spin-$\frac{1}{2}$ fields need to be restricted to suitable ideals to guarantee closure under transformation. A spinor space is an ideal of the spacetime algebra. In this paper the generalized bivectors form a subspace but are not required to form an ideal.

2. Clifford spacetime algebra

In this section we construct the two Clifford algebras that represent spacetime and some of the important properties of these algebras. Measurement of physical quantities requires a reference frame. A reference frame consists of an origin and a collection of direction elements. We have a single time direction which we denote by $\hat{\mathbf{t}}$ and three spatial directions which we denote by $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. A 4-vector with components (a^0, a^1, a^2, a^3) is given by

$$\mathbf{s} = a_0\hat{\mathbf{t}} + a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}}. \tag{1}$$

Addition is defined componentwise as is multiplication by a real scalar so these vectors form a real four-dimensional vector space. We introduce a bilinear, distributive associative product on vectors that gives the Lorentz metric when any vector is taken with itself. Hence

$$\mathbf{s}^2 = {a_0}^2\hat{\mathbf{t}}^2 + {a_1}^2\hat{\mathbf{x}}^2 + {a_2}^2\hat{\mathbf{y}}^2 + {a_3}^2\hat{\mathbf{z}}^2 + a_0a_1(\hat{\mathbf{x}}\hat{\mathbf{t}} + \hat{\mathbf{t}}\hat{\mathbf{x}}) + a_0a_2(\hat{\mathbf{y}}\hat{\mathbf{t}} + \hat{\mathbf{t}}\hat{\mathbf{y}}) + a_0a_3(\hat{\mathbf{z}}\hat{\mathbf{t}} + \hat{\mathbf{t}}\hat{\mathbf{z}})$$
$$+a_1a_2(\hat{\mathbf{x}}\hat{\mathbf{y}} + \hat{\mathbf{y}}\hat{\mathbf{x}}) + a_2a_3(\hat{\mathbf{y}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{y}}) + a_3a_1(\hat{\mathbf{z}}\hat{\mathbf{x}} + \hat{\mathbf{x}}\hat{\mathbf{z}}). \tag{2}$$

Taking $\hat{\mathbf{t}}\hat{\mathbf{x}} = -\hat{\mathbf{x}}\hat{\mathbf{t}}$, $\hat{\mathbf{t}}\hat{\mathbf{y}} = -\hat{\mathbf{y}}\hat{\mathbf{t}}$, $\hat{\mathbf{t}}\hat{\mathbf{z}} = -\hat{\mathbf{z}}\hat{\mathbf{t}}$, $\hat{\mathbf{x}}\hat{\mathbf{y}} = -\hat{\mathbf{y}}\hat{\mathbf{x}}$, $\hat{\mathbf{y}}\hat{\mathbf{z}} = -\hat{\mathbf{z}}\hat{\mathbf{y}}$, $\hat{\mathbf{z}}\hat{\mathbf{x}} = -\hat{\mathbf{x}}\hat{\mathbf{z}}$, $\hat{\mathbf{t}}^2 = -\eta$ and $\hat{\mathbf{x}}^2 = \hat{\mathbf{y}}^2 = \hat{\mathbf{z}}^2 = \eta$ where $\eta = \pm 1$ gives the Lorentz metric. That is,

$$\mathbf{s}^2 = \eta(-{a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2). \tag{3}$$

In 4-vector notation $\mathbf{e}_0 = \hat{\mathbf{t}}$, $\mathbf{e}_1 = \hat{\mathbf{x}}$, $\mathbf{e}_2 = \hat{\mathbf{y}}$ and $\mathbf{e}_3 = \hat{\mathbf{z}}$. This product generates a 16-dimensional real vector space with basis $1, \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_0\mathbf{e}_1, \mathbf{e}_0\mathbf{e}_2, \mathbf{e}_0\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_0\mathbf{e}_2\mathbf{e}_3, \mathbf{e}_0\mathbf{e}_3\mathbf{e}_1, \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. The Clifford algebra $Cl(3, 1)$ is generated when $\eta = 1$ corresponding to a Lorentz signature $\{-{+}{+}{+}\}$ and $Cl(1, 3)$ when $\eta = -1$ corresponding to a Lorentz signature $\{+{-}{-}{-}\}$. Both spaces are isomorphic upon complexification through the transformation $\mathbf{e}_\mu \mapsto \mathrm{i}\mathbf{e}_\mu$.

To make the notation simpler we use the Einstein summation convention. Hence our general vector is written $\mathbf{s} = a^\mu \mathbf{e}_\mu$. If we only want to sum over space indices then we will use Roman letters. In Clifford algebra there is no difference between covariant and contravariant vectors, nevertheless, the notation is very useful for keeping track of signs. We define $a_\mu = {\mathbf{e}_\mu}^2 a^\mu$ and $\mathbf{e}^\mu = {\mathbf{e}_\mu}^2 \mathbf{e}_\mu$. A final notational convenience is to take concatenation of indices for products of vectors to imply that all the indices take on different values. For example $\mathbf{e}_{\mu\nu} = \mathbf{e}_\mu \mathbf{e}_\nu$ implies $\mu \neq \nu$. The quarternions are generated by $\hat{\mathbf{i}} = -\eta \mathbf{e}_{23}$, $\hat{\mathbf{j}} = -\eta \mathbf{e}_{31}$ and $\hat{\mathbf{k}} = -\eta \mathbf{e}_{12}$. These are also the infinitesimal generators of rotation in the spacetime algebra.

The product $\mathbf{ab}$ of two multivectors $\mathbf{a}$ and $\mathbf{b}$ is associative but not commutative. In fact, it is made up of a commutative inner product $\mathbf{a} \cdot \mathbf{b}$ and an anti-commutative outer product $\mathbf{a} \wedge \mathbf{b}$. They are given in terms of the Clifford product as

$$\mathbf{a} \cdot \mathbf{b} = \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba}) \tag{4}$$

$$\mathbf{a} \wedge \mathbf{b} = \tfrac{1}{2}(\mathbf{ab} - \mathbf{ba}) \tag{5}$$

and the Clifford product decomposes as $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$. The inner and outer products are both non-associative. The outer product satisfies the Jacobi identity:

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) + \mathbf{b} \wedge (\mathbf{c} \wedge \mathbf{a}) + \mathbf{c} \wedge (\mathbf{a} \wedge \mathbf{b}) = 0. \tag{6}$$

On restriction to linear combinations of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ the inner product is the dot product and the outer product is equivalent to the cross product.

We define the following differential operator on spacetime algebra which behaves like a combined grad, curl and div operator:

$$\begin{aligned}\mathbf{d} &= \hat{\mathbf{t}}\partial^t + \hat{\mathbf{x}}\partial^x + \hat{\mathbf{y}}\partial^y + \hat{\mathbf{z}}\partial^z \\ &= \mathbf{e}_\mu \partial^\mu.\end{aligned} \tag{7}$$

It has a number of important properties that we need. When applied to itself we obtain $\mathbf{d}^2 = \mathbf{d} \cdot \mathbf{d} + \mathbf{d} \wedge \mathbf{d}$ where $\mathbf{d} \cdot \mathbf{d} = -\eta \,\Box = -\eta({\partial_t}^2 - \nabla^2)$ is the d'Alembertian operator. The anti-commutative part $\mathbf{d} \wedge \mathbf{d} = 0$ if and only if $\partial^\mu \partial^\nu = \partial^\nu \partial^\mu$. In other words, $\mathbf{d} \wedge \mathbf{d} = 0$ on continuously differentiable functions and consequently $\mathbf{d}^2 = -\eta \,\Box$ is a scalar. If $\mathbf{d}^2$ is a scalar then for any multivector $\mathbf{A}$ we have

$$\mathbf{d} \cdot (\mathbf{d} \wedge \mathbf{A}) = 0 \qquad \text{and} \qquad \mathbf{d} \wedge (\mathbf{d} \cdot \mathbf{A}) = 0. \tag{8}$$

These identities encapsulate the well known identities div curl $= 0$ and curl grad $= 0$.

The quadvector $* = \eta \mathbf{e}_{0123}$ gives a pseudo-duality on spacetime algebra: a multi-vector $\mathbf{A}$ is taken to its pseudo-dual $*\mathbf{A}$. This is not a true duality because $*^2 = -1$. The lack of a natural duality is a reflection of the 4π spinorial nature of spacetime.

3. Adding an electromagnetic vector potential to spacetime

We take as our starting point a (electromagnetic) vector potential which we believe is more fundamental than the corresponding (electromagnetic) field tensor. This has been demonstrated by the Aharonov–Bohm effect [1, 9] where a vector potential has been shown to give a topological phase shift in a region of space where the electric and magnetic fields are zero.

Let $\mathbf{A} = \mathbf{A}^{\mu}\mathbf{e}_{\mu}$ be the vector potential. Then

$$\mathbf{dA} = \mathbf{d} \cdot \mathbf{A} + \mathbf{d} \wedge \mathbf{A} \tag{9}$$

is the sum of a scalar part and a bivector part. The scalar part is a gauge term $\mathbf{L} = \mathbf{d} \cdot \mathbf{A} = \partial^{\mu} A_{\mu}$ The bivector part is equivalent to a rank-2 anti-symmetric (electromagnetic) field tensor and is given by $\mathbf{F} = \frac{1}{2} F^{\mu\nu} \mathbf{e}_{\mu\nu}$ where $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$. Hence any vector potential gives rise to a gauge scalar $\mathbf{L} = \mathbf{d} \cdot \mathbf{A}$ and a bivector field $\mathbf{F} = \mathbf{d} \wedge \mathbf{A}$.

4. The electromagnetic bivector field

The electromagnetic bivector field is anti-symmetric and its components are those of the electromagnetic field tensor. The $\mathbf{e}_{0k}$ bivector components give the electric field as $E^k = F^{0k}$ and the space bivector components give the magnetic field as $B^k = \frac{1}{2} \epsilon^{k}{}_{pq} F^{pq}$. Given a vector current $\mathbf{J} = J^{\mu} \mathbf{e}_{\mu}$, we demonstrate the Maxwell equations are $\mathbf{dF} = \mathbf{J}$. Since $\mathbf{d} \wedge \mathbf{F}$ is a vector and $\mathbf{d} \cdot \mathbf{F}$ is a trivector then this equation decomposes into a homogeneous part $\mathbf{d} \cdot \mathbf{F} = 0$ and an inhomogeneous part $\mathbf{d} \wedge \mathbf{F} = \mathbf{J}$. The homogeneous Maxwell equations are the components of the trivector terms given by $\mathbf{d} \cdot \mathbf{F} = 0$. The inhomogeneous Maxwell equations are the components of the vector terms given by $\mathbf{d} \wedge \mathbf{F} = \mathbf{J}$.

If the electromagnetic bivector field is given by a vector potential then $\mathbf{F} = \mathbf{d} \wedge \mathbf{A}$ and hence $\mathbf{d} \cdot \mathbf{F} = \mathbf{d} \cdot (\mathbf{d} \wedge \mathbf{A}) = 0$. In other words, $\mathbf{F}$ arising from a vector potential implies there is no magnetic charge. This is a well known property of the Maxwell equations. Nevertheless, this does not preclude the existence of magnetic monopoles on a non-trivial topology, such as the Dirac monopole [3].

Since $\mathbf{J}$ is a vector then $\mathbf{d} \cdot \mathbf{J} = \partial_{\mu} J^{\mu}$ which when zero gives conservation of electric charge. The Maxwell equations guarantee conservation of current because $\mathbf{d} \cdot \mathbf{J} = \mathbf{d} \cdot (\mathbf{d} \wedge \mathbf{F}) = 0$.

5. Gauge freedom

A general gauge transformation of the electromagnetic potential is

$$\mathbf{A} \mapsto \mathbf{A}' = \mathbf{A} + \mathbf{d}\chi. \tag{10}$$

This transformation leaves the electromagnetic bivector field unchanged, thus $\mathbf{F}' = \mathbf{d} \wedge (\mathbf{A} + \mathbf{d} \cdot \chi) = \mathbf{d} \wedge \mathbf{A} + \mathbf{d} \wedge (\mathbf{d} \cdot \chi) = \mathbf{d} \wedge \mathbf{A} = \mathbf{F}$. The gauge scalar becomes $\mathbf{L}' = \mathbf{L} + \mathbf{d}^2\chi$. If one chooses χ such that $\mathbf{L} = -\mathbf{d}^2\chi$ then the gauge scalar vanishes and one has transformed to the Lorentz gauge. The Maxwell equations for the vector potential are given by

$$\mathbf{d} \wedge (\mathbf{d} \wedge \mathbf{A}) = \mathbf{J}. \tag{11}$$

If we impose the Lorentz gauge condition, that is, $\mathbf{L} = \mathbf{d} \cdot \mathbf{A} = 0$ then

$$\mathbf{d}^2\mathbf{A} = \mathbf{J}. \tag{12}$$

6. Adding mass: the Proca equation

The Proca equation is a massive electromagnetic vector potential. Adding a mass term to equation (12) gives $\mathbf{d} \wedge (\mathbf{d} \wedge \mathbf{A}) - \eta m^2 \mathbf{A} = \mathbf{J}$. Current conservation requires $\mathbf{d} \cdot \mathbf{J} = \mathbf{d} \cdot (\mathbf{d} \wedge (\mathbf{d} \wedge \mathbf{A})) - \eta m^2 \mathbf{d} \cdot \mathbf{A} = -\eta m^2 \mathbf{L} = 0$. Hence the vanishing of the gauge scalar is a necessary condition for current conservation. The Proca equation is

$$(\mathbf{d}^2 - \eta m^2)\mathbf{A} = \mathbf{J} \qquad \mathbf{L} = 0. \tag{13}$$

This requires the Lorentz gauge condition. In other words, the Proca equation is not gauge invariant.

7. The Dirac algebra

The full Dirac algebra is taken to be the complexification of $Cl(3, 1)$ or $Cl(1, 3)$ depending on the choice of metric (or η). Denote these choices by $D(3, 1)$ and $D(1, 3)$, respectively. The complexifications of $Cl(3, 1)$ and $Cl(1, 3)$ are isomorphic according to the transformation $\mathbf{e}_\mu \mapsto \mathbf{i}\mathbf{e}_\mu$. This algebra may also be thought of as the real Clifford algebra generated by the vectors $\mathbf{e}_0, \mathbf{i}\mathbf{e}_0, \mathbf{e}_1, \mathbf{i}\mathbf{e}_1, \mathbf{e}_2, \mathbf{i}\mathbf{e}_2, \mathbf{e}_3, \mathbf{i}\mathbf{e}_3$. Let $D(4)$ denote the full Dirac algebra for a given choice of η. Hence $D(4) \cong Cl(1, 3) \oplus Cl(3, 1)$ as vector spaces. If $\eta = 1$ then $Cl(3, 1)$ is the real part of $D(3, 1)$ and $Cl(1, 3)$ the imaginary part of $D(1, 3)$. The reverse being true for $\eta = -1$.

The full Dirac algebra has an important decomposition. Define $\langle D(4)\rangle_0$ to be the scalar part of $D(4)$ generated by 1; $\langle D_4\rangle_1$ to be the vector part of $D(4)$ generated by $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$; $\langle D(4)\rangle_2$ to be the bivector part of $D(4)$ generated by $\mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03}, \mathbf{e}_{23}, \mathbf{e}_{31}$ and $\mathbf{e}_{12}$; $\langle D(4)\rangle_3$ to be the trivector part of $D(4)$ generated by $\mathbf{e}_{123}$, $\mathbf{e}_{023}$, $\mathbf{e}_{031}$ and $\mathbf{e}_{012}$; and $\langle D(4)\rangle_4$ to be the pseudo-scalar part of $D(4)$ generated by $*$. These subspaces decompose the full Dirac algebra into the direct sum $D(4) \cong \langle D(4)\rangle_0 \oplus \langle D(4)\rangle_1 \oplus \langle D(4)\rangle_2 \oplus \langle D(4)\rangle_3 \oplus \langle D(4)\rangle_4$. In particular, one can decompose an element $\mathbf{u}$ of $D(4)$ thus

$$\mathbf{u} = \langle \mathbf{u}\rangle_0 + \langle \mathbf{u}\rangle_1 + \langle \mathbf{u}\rangle_2 + \langle \mathbf{u}\rangle_3 + \langle \mathbf{u}\rangle_4 \tag{14}$$

where $\langle u\rangle_n \in \langle D(4)\rangle_n$. The concept of reversion amounts to reversing the order of vectors constituting a given multi-vector. Formally, one defines the reversion of $\mathbf{u}$ to be $\tilde{\mathbf{u}} = \langle \mathbf{u}\rangle_0 + \langle \mathbf{u}\rangle_1 - \langle \mathbf{u}\rangle_2 - \langle \mathbf{u}\rangle_3 + \langle \mathbf{u}\rangle_4$.

The full Dirac algebra is a 16-dimensional complex Clifford algebra. In the next section the Dirac field is constructed in terms from the scalar, bivectors and the pseudo-scalar. Call the algebra spanned by these Clifford elements the (restricted) Dirac algebra. Generators for this algebra are given by $\mathbf{e}_{0k}$ and hence the Dirac algebra is isomorphic to the complexified Clifford algebra $Cl(3, 0)$. Hence we denote the Dirac algebra by $D(3)$ and call its elements generalized bivectors.

8. Construction of the Dirac equation

In relativistic quantum mechanics Einstein's law of mass–energy is given by the Klein–Gordon equation. The Klein–Gordon operator in Clifford spacetime is given by $\mathbf{d}^2 - \eta m^2$. This operator is a scalar and consequently can act on any multi-vector field. Conservation of mass–energy through the correspondence principle is given by the Klein–Gordon equation thus

$$(\mathbf{d}^2 - \eta m^2)\varphi = 0. \tag{15}$$

Hence any special relativistic quantum mechanical equation must imply the Klein–Gordon equation.

We seek to construct a Dirac equation of the form

$$\mathbf{D}\boldsymbol{\psi} = m\boldsymbol{\psi}\mathbf{k} \tag{16}$$

where $\mathbf{D}$ is a Clifford linear differential operator and $\mathbf{k}$ is a fixed Clifford element. This equation will be constructed according to the following list of assumptions:

(a) $\boldsymbol{\psi}$ satisfies the Klein–Gordon equation,
(b) $\boldsymbol{\psi}$ is a generalized bivector field,
(c) there are no preferred directions with respect to the space vectors $\mathbf{e}_k$,
(d) the linear operator $\mathbf{D}$ is independent of spacetime position.

Applying $\mathbf{D}$ to (16) gives

$$\mathbf{D}^2\psi = \mathbf{D}(m\psi\mathbf{k}) = m^2\psi\mathbf{k}^2. \tag{17}$$

By assumption (a) and with out loss of generality we can assume $\mathbf{D}^2 = \mathbf{d}^2$ and $\mathbf{k}^2 = \eta$. Since $\mathbf{D}$ is a linear differential operator it can be written $\mathbf{D} = \mathbf{a}_\mu\partial^\mu$ and by assumption (d) the $\mathbf{a}_\mu$ are fixed Clifford elements. Hence

$$\mathbf{D}^2 = \sum_\mu \mathbf{a}_\mu{}^2\partial^{\mu 2} + \sum_{\mu>\nu}(\mathbf{a}_\mu\mathbf{a}_\nu + \mathbf{a}_\nu\mathbf{a}_\mu)\partial^\mu\partial^\nu \tag{18}$$

and so $\mathbf{D}^2 = \mathbf{d}^2$ if and only if $\mathbf{a}_\mu{}^2 = \mathbf{e}_\mu{}^2$ and $\mathbf{a}_\mu\mathbf{a}_\nu + \mathbf{a}_\nu\mathbf{a}_\mu = 0$. One solution to this equation is $\mathbf{a}_\mu = \mathbf{e}_\mu$. For this choice $\mathbf{D} = \mathbf{d}$. There are no other choices without violating assumption (c).

The only choices of $\mathbf{k}$ consistent with assumption (c) are: ± 1, $\pm\mathrm{i}*$, $\pm\mathrm{i}\mathbf{e}_0$ and $\pm\mathrm{i}\mathbf{e}_{123}$ for $\eta = +1$ and $\pm\mathrm{i}$, $\pm *$, $\pm\mathrm{i}\mathbf{e}_0$ and $\pm\mathrm{i}\mathbf{e}_{123}$ for $\eta = -1$. The Dirac–Hestenes equation [6] makes the choice $\mathbf{k} = \mathbf{e}_{012}$ violating assumption (c) and thus giving a special significance to $\mathbf{e}_3$, suggesting that the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions are intrinsically different from the $\mathbf{e}_3$ direction. If $\mathbf{k}$ is the scalar or pseudo-scalar choice the only solution to $\mathbf{d}\psi = m\psi\mathbf{k}$ is $\psi = 0$. To get a non-trivial theory $\mathbf{k}$ must be $\pm\mathrm{i}\mathbf{e}_0$ or $\pm\mathrm{i}\mathbf{e}_{123}$. We then have four possible choices for $\mathbf{k}$. Any generalized bivector can be written as $\psi = \mathbf{e}_0\psi_1 + \mathbf{e}_{123}\psi_2$, where ψ_1 and ψ_2 are vectors. Hence the different choices of $\mathbf{k}$ provide alternative but equivalent ways for representing ψ. In this paper we make the choice $\mathbf{k} = -\mathrm{i}\mathbf{e}_0$ and the Dirac equation is

$$\mathrm{i}\mathbf{d}\psi = m\psi\mathbf{e}_0. \tag{19}$$

One may think that this equation singles out the time direction. However, this is just a representation of Einstein's mass–energy equivalence; that is, both the energy operator $\mathrm{i}\partial^0$ and the rest energy (or mass) term appear on the time vector $\mathbf{e}_0$.

9. The Dirac current

Define the Dirac adjoint of ψ to be $\psi^\dagger = \tilde{\psi}^*$. A generalized bivector can be written as $\psi = \mathbf{e}_0\psi_1 + \mathbf{e}_{123}\psi_2$ where ψ_1 and ψ_2 are vectors. Hence $\psi^\dagger = \psi_1^*\mathbf{e}_0 - \psi_2^*\mathbf{e}_{123}$. The Dirac adjoint of ψ satisfies

$$\mathrm{i}\psi^\dagger\mathbf{d}^\dagger = -m\mathbf{e}_0\psi^\dagger \tag{20}$$

with the convention that $\mathbf{d}^\dagger$ acts to the left. Note that $\mathbf{d}$ and $\mathbf{d}^\dagger$ are identical except for their direction of action. Define the Dirac conjugate of ψ by $\bar{\psi} = -\eta\mathbf{e}_0\psi^\dagger\mathbf{e}_0$. Hence $\bar{\psi} = \mathbf{e}_0\psi_1^* - \mathbf{e}_{123}\psi_2^*$. From this one constructs the following positive-definite quantity:

$$\langle\bar{\psi}\psi\rangle_0 = \sum_\mu |\psi_1{}^\mu|^2 + \sum_\mu |\psi_2{}^\mu|^2. \tag{21}$$

The Dirac current is defined to be the vector $\mathbf{J} = J^\mu\mathbf{e}_\mu$ where $J_\mu = \langle\mathbf{e}_0\psi^\dagger\mathbf{e}_\mu\psi\rangle_0$. This current is conserved because $\mathbf{d}\cdot\mathbf{J} = \langle\mathbf{e}_0(\psi^\dagger\mathbf{d}^\dagger)\psi\rangle_0 + \langle\mathbf{e}_0\psi^\dagger(\mathbf{d}\psi)\rangle_0 = -\mathrm{i}m\eta\langle\psi^\dagger\psi\rangle_0 + \mathrm{i}m\eta\langle\psi^\dagger\psi\rangle_0 = 0$.

10. Plane-wave solutions to the Dirac equation

We seek a plane-wave solution propagating along the x-axis of the form $\psi = \mathbf{A}\mathrm{e}^{\mathrm{i}(\omega t - kx)}$. Substituting into the Dirac equation one obtains

$$-(\omega\mathbf{e}_0 - k\mathbf{e}_1)\mathbf{A}\mathbf{e}_0 = m\mathbf{A}. \tag{22}$$

Suppose $\mathbf{A} = a + b\mathbf{e}_{23} + c\mathbf{e}_{0123} + d\mathbf{e}_{01}$ then

$$-(\omega\mathbf{e}_0 - k\mathbf{e}_1)(a\mathbf{e}_0 + b\mathbf{e}_{023} + \eta c\mathbf{e}_{123} + \eta d\mathbf{e}_1) - m\mathbf{A} = 0$$

$$\eta\omega a + \eta\omega b\mathbf{e}_{23} - \eta\omega c\mathbf{e}_{0123} - \eta\omega d\mathbf{e}_{01} - ka\mathbf{e}_{01} - kb\mathbf{e}_{0123} + kc\mathbf{e}_{23} + kd$$
$$-ma - mb\mathbf{e}_{23} - mc\mathbf{e}_{0123} - md\mathbf{e}_{01} = 0.$$

Collecting like terms gives

$$\begin{aligned} &\eta\omega a + kd - ma = 0 \\ &-\eta\omega d - ka - md = 0 \\ &\eta\omega b + kc - mb = 0 \\ &-\eta\omega c - kb - mc = 0. \end{aligned} \tag{23}$$

This gives the following system of homogeneous equations to solve:

$$\begin{bmatrix} \eta\omega - m & k & 0 & 0 \\ -k & -\eta\omega - m & 0 & 0 \\ 0 & 0 & \eta\omega - m & k \\ 0 & 0 & -k & -\eta\omega - m \end{bmatrix} \begin{bmatrix} a \\ d \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{24}$$

This system has a non-trivial solution if and only if the determinant vanishes; that is, $(-\omega^2 + m^2 + k^2)^2 = 0$ or $\omega = \pm\sqrt{k^2 + m^2}$. The general solution to this system is

$$\begin{bmatrix} a \\ d \\ b \\ c \end{bmatrix} = \begin{bmatrix} -(\eta\omega + m)a_1 \\ ka_1 \\ (\eta\omega + m)a_2 \\ -ka_2 \end{bmatrix}. \tag{25}$$

Hence

$$\begin{aligned} \mathbf{A} &= a_1(-(\eta\omega + m) + k\mathbf{e}_{01}) - a_2(k\mathbf{e}_{0123} - (\eta\omega + m)\mathbf{e}_{23}) \\ &= a_1\mathbf{e}_0((\omega + \eta m)\mathbf{e}_0 + k\mathbf{e}_{01}) + a_2\mathbf{e}_{123}(k\mathbf{e}_0 - (\eta\omega + m)\mathbf{e}_1). \end{aligned} \tag{26}$$

The other possibility for $\mathbf{A}$ is to take $\mathbf{A} = a\mathbf{e}_{02} + b\mathbf{e}_{03} + c\mathbf{e}_{31} + d\mathbf{e}_{12}$. Substituting into the Dirac equation gives

$$(-\omega\mathbf{e}_0 + k\mathbf{e}_1)(a\eta\mathbf{e}_2 + b\eta\mathbf{e}_3 + c\mathbf{e}_{031} + d\mathbf{e}_{012}) - m\mathbf{A} = 0$$

$$-\eta\omega a\mathbf{e}_{02} - \eta\omega b\mathbf{e}_{03} + \eta\omega c\mathbf{e}_{31} + \eta\omega d\mathbf{e}_{12} + \eta ka\mathbf{e}_{12} - \eta kb\mathbf{e}_{31} + \eta kc\mathbf{e}_{03}$$
$$-\eta kd\mathbf{e}_{02} - ma\mathbf{e}_{02} - mb\mathbf{e}_{03} - mc\mathbf{e}_{31} - md\mathbf{e}_{12} = 0.$$

Collecting like terms yields

$$\begin{aligned} &-\eta\omega a - \eta kd - ma = 0 \\ &\eta\omega d + \eta ka - md = 0 \\ &-\eta\omega b + \eta kc - mb = 0 \\ &\eta\omega c - \eta kb - mc = 0. \end{aligned} \tag{27}$$

Thus we have the following homogeneous system of linear equations to solve:

$$\begin{bmatrix} -\eta\omega - m & -\eta k & 0 & 0 \\ \eta k & \eta\omega - m & 0 & 0 \\ 0 & 0 & -\eta\omega - m & +\eta k \\ 0 & 0 & -\eta k & \eta\omega - m \end{bmatrix} \begin{bmatrix} a \\ d \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{28}$$

This system has non-trivial solutions if and only if the determinant, which is $(-\omega^2+m^2+k^2)^2$, vanishes. This occurs exactly when $\omega = \pm\sqrt{k^2+m^2}$. The general solution is

$$\begin{bmatrix} a \\ d \\ b \\ c \end{bmatrix} = \begin{bmatrix} ka_3 \\ -(\omega+\eta m)a_3 \\ ka_4 \\ (\omega+\eta m)a_4 \end{bmatrix}. \tag{29}$$

Hence

$$\begin{aligned} \mathbf{A} &= a_3(k\mathbf{e}_{02} - (\omega+\eta m)\mathbf{e}_{12}) + a_4(k\mathbf{e}_{03} + (\omega+\eta m)\mathbf{e}_{31}) \\ &= k\mathbf{e}_0(a_3\mathbf{e}_2 + a_4\mathbf{e}_3) + (\eta\omega+m)\mathbf{e}_{123}(a_4\mathbf{e}_2 - a_3\mathbf{e}_3). \end{aligned} \tag{30}$$

The most general plane-wave solution propagating along the x-axis is

$$\begin{aligned} \psi = \{&\mathbf{e}_0[a_1((\omega+\eta m)\mathbf{e}_0 + k\mathbf{e}_1) + k(a_3\mathbf{e}_2 + a_4\mathbf{e}_3)] \\ &+\mathbf{e}_{123}[a_2(k\mathbf{e}_0 - (\eta\omega+m)\mathbf{e}_1) + (\eta\omega+m)(a_4\mathbf{e}_2 - a_3\mathbf{e}_3)]\}\mathrm{e}^{\mathrm{i}(\omega t - kx)}. \end{aligned} \tag{31}$$

Considering that ω can be positive or negative we have eight independent solutions instead of the usual four. The interpretation of these solutions will be dealt with in a later section. The norm of this solution squared is by (21)

$$|\psi|^2 = 2\omega(\omega+\eta m)(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2). \tag{32}$$

11. Electromagnetic coupling to the Dirac equation

In this section an electromagnetic vector potential $\mathbf{A}$ is coupled to a Dirac generalized bivector field ψ using the minimal coupling condition. Namely, one replaces $\mathbf{d}$ with $\mathbf{d} + \mathrm{i}q\mathbf{A}$. This substitution gives the minimally coupled Dirac equation

$$(\mathrm{i}\mathbf{d} - q\mathbf{A})\psi = m\psi\mathbf{e}_0. \tag{33}$$

Hence the minimally coupled Klein–Gordon equation is

$$(\mathrm{i}\mathbf{d} - q\mathbf{A})^2\psi + \eta m^2\psi = 0. \tag{34}$$

Expanding the brackets one obtains $(\mathrm{i}\mathbf{d} - q\mathbf{A})^2\psi = -\mathbf{d}^2\psi + q^2\mathbf{A}^2\psi - 2\mathrm{i}q(\mathbf{A}\cdot\mathbf{d})\psi - \mathrm{i}q\mathbf{L}\psi - \mathrm{i}q\mathbf{F}\psi$. The fourth term vanishes when the Lorentz gauge is imposed; $\mathbf{L} = \mathbf{d}\cdot\mathbf{A} = 0$. The expanded form of the minimally coupled Klein–Gordon equation is

$$[-\mathbf{d}^2 + \eta m^2 + q^2\mathbf{A}^2 - \mathrm{i}q(2\mathbf{A}\cdot\mathbf{d} + \mathbf{L})]\psi - \mathrm{i}q\mathbf{F}\psi = 0. \tag{35}$$

The operator in the square brackets is a complex scalar operator. The last term provides the spin interaction with the electromagnetic field. Note that for the coupled Dirac equation the Dirac current is still conserved. A gauge transformation of the electromagnetic vector potential $\mathbf{A} \mapsto \mathbf{A} + \mathbf{d}\chi$ leaves the Dirac equation invariant if the Dirac field ψ undergoes a gauge transformation $\psi \mapsto \mathrm{e}^{-\mathrm{i}q\chi}\psi$.

12. Clifford fields on geometric algebra

Let I be the set of all numerically ordered sequences including the null sequence with out repetition from the set $\{0, 1, 2, 3\}$. The set I is finite and serves to label the standard multivector basis. Hence any $\psi \in D(4)$ can be written as

$$\psi = \sum_{\alpha\in I} \psi^\alpha \mathbf{e}_\alpha \tag{36}$$

where each $\psi^\alpha \in \mathbb{C}$. One can restrict to the Dirac algebra $D(3)$ by further restricting the set I to those elements of even length. Given $\alpha \in I$ define $|\alpha|$ to be the length of the sequence. Define the conjugate of $\psi \in D(4)$ by

$$\bar{\psi} = \sum_{\alpha \in I} (-\eta)^{|\alpha|+1} \psi^{\alpha *} \mathbf{e}_0 \tilde{\mathbf{e}}_\alpha \mathbf{e}_0. \tag{37}$$

If $\eta = -1$ or whenever no part of ψ coincides with $D(3)$ then $\bar{\psi} = \mathbf{e}_0 \tilde{\psi}^* \mathbf{e}_0$. Conjugation has the property that $\overline{\psi\varphi} = \bar{\varphi}\bar{\psi}$.

An inner product on $D(4)$ is given by $\langle \psi | \varphi \rangle = \langle \bar{\psi} \varphi \rangle_0$ for all $\psi, \varphi \in D(4)$. Clearly, it is conjugate-linear in ψ, linear in φ and $\langle \bar{\psi} \varphi \rangle_0^* = \sum_{\alpha \in I} \langle \psi^{\alpha *} \bar{\mathbf{e}}_\alpha \varphi^\alpha \mathbf{e}_\alpha \rangle_0^* = \sum_{\alpha \in I} \langle \varphi^{\alpha *} \bar{\mathbf{e}}_\alpha \psi^\alpha \mathbf{e}_\alpha \rangle_0 = \langle \bar{\varphi} \psi \rangle_0$. Moreover, the norm is given by

$$\begin{aligned} |\psi|^2 &= \langle \bar{\psi} \psi \rangle_0 \\ &= \sum_{\alpha \in I} \langle \overline{\psi^\alpha \mathbf{e}_\alpha} \psi^\alpha \mathbf{e}_\alpha \rangle_0 \\ &= \sum_{\alpha \in I} (-\eta)^{|\alpha|+1} |\psi^\alpha|^2 \mathbf{e}_0 \tilde{\mathbf{e}}_\alpha \mathbf{e}_0 \mathbf{e}_\alpha \\ &= \sum_{\alpha \in I} |\psi^\alpha|^2 \end{aligned}$$

with the last line being because $\mathbf{e}_0 \tilde{\mathbf{e}}_\alpha \mathbf{e}_0 \mathbf{e}_\alpha = (-\eta)^{|\alpha|+1}$.

A Clifford field is defined to be a smooth map $\psi : \mathbb{R}^4 \to D(4)$ such that

$$\int_{\mathbb{R}^4} \langle \bar{\psi(\mathbf{x})} \psi(\mathbf{x}) \rangle_0 \, \mathrm{d}^4 x < \infty \tag{38}$$

where $\mathbf{x} = x^\mu \mathbf{e}_\mu$ denotes $(x^\mu) \in \mathbb{R}^4$. Denote the Hilbert space of Clifford fields by $D_2^\infty(4)$. The inner product for this space is

$$\langle \psi | \varphi \rangle = \int_{\mathbb{R}^4} \langle \bar{\psi}(\mathbf{x}) \varphi(\mathbf{x}) \rangle_0 \, \mathrm{d}^4 x \tag{39}$$

for all $\psi, \varphi \in D_2^\infty(4)$. The norm squared is

$$|\psi|^2 = \sum_{\alpha \in I} \int_{\mathbb{R}^4} |\psi^\alpha(\mathbf{x})|^2 \, \mathrm{d}^4 x \tag{40}$$

where $\psi(\mathbf{x}) = \sum_\alpha \psi^\alpha \mathbf{e}_\alpha \in D_2^\infty(4)$.

Every Clifford linear operator $\mathbf{A}$ can be decomposed as

$$\mathbf{A} = \sum_{\alpha \in I} \mathbf{e}_\alpha (A^\alpha + \mathrm{i}\mathbf{B}^\alpha) \tag{41}$$

where A^α and $\mathbf{B}^\alpha$ are real linear operators. The conjugate is defined by

$$\bar{\mathbf{A}} = \sum_{\alpha \in I} \bar{\mathbf{e}}_\alpha (A^\alpha - \mathrm{i}\mathbf{B}^\alpha). \tag{42}$$

Note that the operator $\mathbf{U} = \mathrm{e}^{\mathbf{A}}$ satisfies $\bar{\mathbf{U}} = \mathrm{e}^{\bar{\mathbf{A}}}$.

13. Spacetime transformations

This section collects together the transformations for translations, boosts, spatial reflections, rotations, translations, parity, time reversal and inversion in Clifford spacetime.

Translation of a vector $\mathbf{x}$ by a vector $\mathbf{a}$ is given by $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$. Translation has a different form to the others because it is inhomogeneous. A general homogeneous transformation that preserves the scalar is of the form

$$\mathbf{x} \mapsto \mathbf{U}\mathbf{x}\mathbf{U}^{-1} \tag{43}$$

for some invertible $\mathbf{U}$. Note that such transformations maintain or reverse the sign of the pseudo-scalar. Two important properties of these type of transformations are: given two multi-vectors $\mathbf{x}$ and $\mathbf{y}$ transforming according to $\mathbf{x} \mapsto \mathbf{x}' = \mathbf{U}\mathbf{x}\mathbf{U}^{-1}$ and $\mathbf{y} \mapsto \mathbf{y}' = \mathbf{U}\mathbf{y}\mathbf{U}^{-1}$ then

(a) $\mathbf{x}\mathbf{y} \mapsto \mathbf{U}\mathbf{x}\mathbf{y}\mathbf{U}^{-1}$,
(b) the inverse transformation is $\mathbf{x}' \mapsto \mathbf{x} = \mathbf{U}^{-1}\mathbf{x}'\mathbf{U}$.

The specific form of the important transformations are given in the following list.

- Translation by a vector $\mathbf{a}$:

$$\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}. \tag{44}$$

- Rotation anticlockwise about an axis $\mathbf{n} = n^1\hat{\mathbf{i}} + n^2\hat{\mathbf{j}} + n^3\hat{\mathbf{k}}$ where $\mathbf{n}^2 = -1$ by an angle θ:

$$\mathbf{x} \mapsto \mathrm{e}^{\frac{1}{2}\mathbf{n}\theta}\mathbf{x}\mathrm{e}^{-\frac{1}{2}\mathbf{n}\theta}. \tag{45}$$

- Boost in the direction $\mathbf{m} = m^k\mathbf{e}_{0k}$ (where $\mathbf{m}^2 = 1$) with speed v or rapidity $\varphi = \tanh^{-1}(\frac{v}{c})$:

$$\mathbf{x} \mapsto \mathrm{e}^{\frac{1}{2}\mathbf{m}\varphi}\mathbf{x}\mathrm{e}^{-\frac{1}{2}\mathbf{m}\varphi}. \tag{46}$$

- Spatial reflection through the space plane perpendicular to the vector $\mathbf{n} = n^k\mathbf{e}_k$:

$$\mathbf{x} \mapsto (*\mathbf{n})\mathbf{x}(*\mathbf{n})^{-1}. \tag{47}$$

- Parity (P):

$$\mathbf{x} \mapsto P\mathbf{x} = -\eta\mathbf{e}_0\mathbf{x}\mathbf{e}_0. \tag{48}$$

- Time reversal (T):

$$\mathbf{x} \mapsto T\mathbf{x} = -\eta\mathbf{e}_{123}\mathbf{x}\mathbf{e}_{123}. \tag{49}$$

- Inversion (I):

$$\mathbf{x} \mapsto I\mathbf{x} = -*\mathbf{x}*. \tag{50}$$

The last three transformations satisfy $IPT\mathbf{x} = \mathbf{x}$. We call this result the *IPT* theorem. It is the precursor to the *CPT* theorem.

14. Transformation of equations

The Dirac, Maxwell and Klein–Gordon equations involve the operator $\mathbf{d}$. The first step to describing transformations of these equations is, therefore, to determine how $\mathbf{d}$ transforms. Clearly, $\mathbf{d}$ is invariant under translations. Suppose then that we are given a homogeneous transformation $\mathbf{x} \mapsto \mathbf{x}' = \mathbf{U}\mathbf{x}\mathbf{U}^{-1}$. Since this transformation is linear in $\mathbf{x}$ we can write $x'^{\mu} = R^{\mu}{}_{\nu}x^{\nu}$ then $\partial'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\partial^{\nu} = R^{\mu}{}_{\nu}\partial^{\nu}$. Hence $\mathbf{d}' = \mathbf{U}\mathbf{d}\mathbf{U}^{-1}$.

The Maxwell equations are invariant under the transformation

$$\begin{aligned}
&\mathbf{A} \mapsto \mathbf{A}' = \mathbf{U}\mathbf{A}\mathbf{U}^{-1}\\
&\mathbf{F} \mapsto \mathbf{F}' = \mathbf{U}\mathbf{F}\mathbf{U}^{-1}\\
&\mathbf{J} \mapsto \mathbf{J}' = \mathbf{U}\mathbf{J}\mathbf{U}^{-1}.
\end{aligned} \tag{51}$$

We have $\mathbf{F}' = \mathbf{U}\mathbf{d}\mathbf{A}\mathbf{U}^{-1} = \mathbf{U}\mathbf{d}\mathbf{U}^{-1}\mathbf{U}\mathbf{A}\mathbf{U}^{-1} = \mathbf{d}'\mathbf{A}'$ and $\mathbf{d}'\mathbf{F}' = (\mathbf{U}\mathbf{d}\mathbf{U}^{-1})(\mathbf{U}\mathbf{F}\mathbf{U}^{-1}) = \mathbf{U}\mathbf{d}\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}\mathbf{J}\mathbf{U}' = \mathbf{J}'$.

Consider the coupled Dirac equation. If $\mathbf{U}$ is a generalized bivector then the coupled Dirac equation transforms under

$$\begin{aligned}
&\psi \mapsto \psi' = \mathbf{U}\psi\\
&\mathbf{A} \mapsto \mathbf{A}' = \mathbf{U}\mathbf{A}\mathbf{U}^{-1}.
\end{aligned} \tag{52}$$

This transformation is valid for boosts and rotations. We have $(\mathrm{i}\mathbf{d}' - q\mathbf{A}')\psi' = (\mathrm{i}\mathbf{U}\mathbf{d}\mathbf{U}^{-1} - \mathrm{i}a\mathbf{U}\mathbf{A}\mathbf{U}^{-1})\mathbf{U}\psi = \mathbf{U}(\mathrm{i}\mathbf{d} - q\mathbf{A})\psi = m\mathbf{U}\psi\mathbf{e}_0 = m\psi'\mathbf{e}_0$. When $\mathbf{U}$ is a vector or trivector combination such as for spatial reflections, parity, time reversal and inversion one needs transformations of the form

$$\begin{aligned}
&\psi \mapsto \mathbf{U}\psi\mathbf{U}^{-1}\\
&\mathbf{A} \mapsto \mathbf{U}\mathbf{A}\mathbf{U}^{-1}
\end{aligned} \tag{53}$$

to preserve the generalized bivector nature of ψ. Such a transformation leaves the Dirac equation invariant if and only if $\mathbf{U} \wedge \mathbf{e}_0 = 0$. Of these transformations only spatial reflections and parity satisfy this condition. However, if $\mathbf{U}$ satisfies $\mathbf{U} \cdot \mathbf{e}_0 = 0$ then the Dirac equation is invariant under the transformation

$$\begin{aligned}
&\psi \mapsto \psi' = \mathbf{U}\psi^*\mathbf{U}^{-1}\\
&\mathbf{A} \mapsto \mathbf{A}' = \mathbf{U}\mathbf{A}\mathbf{U}^{-1}\\
&q \mapsto -q.
\end{aligned} \tag{54}$$

This is the form of the transformation for time reversal and inversion. Under this transformation the coupled Dirac equation becomes $(\mathrm{i}\mathbf{d} + q\mathbf{A})\psi^* = -m\psi^*\mathbf{e}_0$; taking the complex conjugate of this one obtains the coupled Dirac equation again.

The coupled Dirac equation has an internal symmetry under the right action $\psi \mapsto \psi\mathbf{U}$ whenever $\mathbf{U} \wedge \mathbf{e}_0 = 0$. Hence it is invariant under any linear combination of $1, \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. These generate the group $U(1) \otimes SU(2)$. In conclusion the Dirac equation has a $U(1) \otimes SU(2)$ global gauge symmetry. The implications of this will be examined in the next paper of the series.

15. Charge conjugation, parity and time reversal

In light of the previous two sections: since $* \cdot \mathbf{e}_0 = 0$ then charge conjugation is given by

$$\psi \mapsto \psi_C = -*\psi^* *. \tag{55}$$

Transforming the Dirac equation followed by complex conjugation gives the equation $(\mathrm{i}\mathbf{d} + q\mathbf{A})\psi_C = m\psi_C\mathbf{e}_0$. Thus ψ_C is the corresponding solution to ψ for opposite charge. Since $\mathbf{e}_0 \wedge \mathbf{e}_0 = 0$ then parity is given by

$$\psi \mapsto \psi_P = -\eta\mathbf{e}_0\psi\mathbf{e}_0. \tag{56}$$

And since $\mathbf{e}_{123} \cdot \mathbf{e}_0 = 0$ then time reversal is given by

$$\psi = \psi_T = -\eta\mathbf{e}_{123}\psi^*\mathbf{e}_{123}. \tag{57}$$

Note that time reversal has the effect of reversing charge. As usual the implication of this is that the anti-particle has the opposite charge to its particle derivative. These transformations satisfy the well known *CPT* theorem: $CPT\psi = -(-\eta)^2 {*}\mathbf{e}_0\mathbf{e}_{123}\psi^{**}\mathbf{e}_{123}\mathbf{e}_0{*} = {*}^2\psi{*}^2 = \psi$.

16. Infinitesimal generators

Infinitesimal generators of transformations play a vital role and are required to preserve the inner product. Consider a one-parameter transformation $\mathbf{R}(\theta)$ with a self-adjoint Clifford linear operator $\mathbf{J}$ as the infinitesimal generator. If the one-parameter group transformation is to preserve the inner product then it must be unitary and so by Stone's theorem of the form $\mathbf{R}(\theta) = \mathrm{e}^{\mathrm{i}\mathbf{J}\theta}$.

Consider a spacetime translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$. We have (by Taylor's theorem) that $\psi(x) \mapsto \psi(\mathbf{x} + \mathbf{a}) = \psi(\mathbf{x}) + \mathbf{a} \cdot \bar{\mathbf{d}}\psi(\mathbf{x}) + \mathrm{O}(a^2)$ where $\bar{\mathbf{d}} = \mathbf{e}_0\mathbf{d}\mathbf{e}_0 = \eta(\mathbf{e}_k\partial^k - \mathbf{e}_0\partial^0)$. The infinitesimal generators of this four-parameter transformation are $-\mathrm{i}\mathbf{e}_\mu \cdot \bar{\mathbf{d}}$. The infinitesimal generator for a spatial translation by $\mathbf{a} = a\hat{\mathbf{a}}$ with $\hat{\mathbf{a}}^2 = \eta$ is $\mathbf{P}_{\hat{\mathbf{a}}} = -\mathrm{i}\eta\hat{\mathbf{a}} \cdot \mathbf{d}$. The momentum operator in the $\mathbf{e}_k$ direction is $\mathbf{P}_k = \mathbf{P}_{\mathbf{e}_k} = -\mathrm{i}\partial^k$. The infinitesimal generator for time translation is $\mathbf{P}_0 = \mathbf{P}_{\mathbf{e}_0} = -\mathrm{i}\partial^0$. To summarize a general translation of $\psi(\mathbf{x})$ by $\mathbf{a} = a^\mu\mathbf{e}_\mu$ is given by $\psi \mapsto \mathrm{e}^{\mathrm{i}a^\mu\mathbf{P}_\mu}\psi$.

Next consider a rotation by θ about an axis $\mathbf{n}$ where $\mathbf{n}^2 = -1$. The infinitesimal generator, $\mathbf{J}_\mathbf{n}$, is derived as follows. We have (by Taylor's theorem)

$$\begin{aligned}\psi(\mathbf{x}) &\mapsto \mathrm{e}^{\frac{1}{2}\mathbf{n}\theta}\psi(\mathrm{e}^{\frac{1}{2}\mathbf{n}\theta}\mathbf{x}\mathrm{e}^{-\frac{1}{2}\mathbf{n}\theta}) \\ &= (1 + \tfrac{1}{2}\mathbf{n}\theta + \mathrm{O}(\theta^2))\psi((1 + \tfrac{1}{2}\mathbf{n}\theta + \mathrm{O}(\theta^2))\mathbf{x}(1 - \tfrac{1}{2}\mathbf{n}\theta + \mathrm{O}(\theta^2))) \\ &= (1 + \theta(\tfrac{1}{2}\mathbf{n} - \eta(\mathbf{n} \wedge \mathbf{x}) \cdot \mathbf{d}) + \mathrm{O}(\theta^2))\psi(\mathbf{x}).\end{aligned} \tag{58}$$

The last line because $\mathbf{n} \wedge \mathbf{x}$ is a spatial vector. Thus the infinitesimal generator is $\mathbf{J}_\mathbf{n} = \mathrm{i}(\eta(\mathbf{n} \wedge \mathbf{x}) \cdot \mathbf{d} - \frac{1}{2}\mathbf{n})$ with $\psi \mapsto \mathrm{e}^{\mathrm{i}\mathbf{J}_\mathbf{n}\theta}\psi$. For example, in the $\mathbf{e}_3$ direction one has $\mathbf{J}_{\hat{\mathbf{k}}} = \mathrm{i}(\eta(\hat{\mathbf{k}} \wedge \mathbf{x}) \cdot \mathbf{d} - \frac{1}{2}\hat{\mathbf{k}}) = \mathrm{i}((x^1\partial^2 - x^2\partial^1) + \frac{1}{2}\eta\mathbf{e}_{12})$. The scalar part is orbital angular momentum. The bivector part is an intrinsic spin term. Define the three Cartesian total angular momentum operators by $\mathbf{J}_k = \mathbf{J}_{-\mathbf{e}_{123}\mathbf{e}_k}$, that is $(\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3) = (\mathbf{J}_{\hat{\mathbf{i}}}, \mathbf{J}_{\hat{\mathbf{j}}}, \mathbf{J}_{\hat{\mathbf{k}}})$. Moreover, one can decompose $\mathbf{J}_k$ into its orbital part and its spin part, thus $\mathbf{J}_k = \mathbf{L}_k + \mathbf{S}_k$ where $\mathbf{L}_k = \mathrm{i}\frac{1}{2}\epsilon_{ijk}(x^i\partial^j - x^j\partial^i)$ and $\mathbf{S}_k = -\mathrm{i}\frac{1}{2}\mathbf{e}_{123}\mathbf{e}_k$. Now the outer product satisfies the conditions of a Lie bracket. Hence the $\mathbf{J}_k$'s form a Lie algebra if we verify closure under the outer product. Since $\mathbf{L}_k$'s are scalar operators and the $\mathbf{S}_{k'}$'s are fixed bivectors, then $\mathbf{L}_k \wedge \mathbf{S}_{k'} = 0$. Thus $\mathbf{J}_i \wedge \mathbf{J}_j = \mathbf{L}_i \wedge \mathbf{L}_j + \mathbf{S}_i \wedge \mathbf{S}_j$ with $\mathbf{L}_i \wedge \mathbf{L}_j = \mathrm{i}\frac{1}{2}\epsilon_{ij}{}^k\mathbf{L}_k$ and $\mathbf{S}_i \wedge \mathbf{S}_j = \mathrm{i}\frac{1}{2}\epsilon_{ij}{}^k\mathbf{S}_k$. Hence the infinitesimal generators of total angular momentum obey

$$\mathbf{J}_i \wedge \mathbf{J}_j = \mathrm{i}\tfrac{1}{2}\epsilon_{ij}{}^k(\mathbf{J}_k) \tag{59}$$

and form the Lie algebra $SU(2)$. Define the total angular momentum squared by $\mathbf{J}^2 = \mathbf{L}^2 + \mathbf{S}^2$, where $\mathbf{L}^2 = \sum_k \mathbf{L}_k{}^2$ and $\mathbf{S}^2 = \sum_k \mathbf{S}_k{}^2 = \frac{1}{2}(\frac{1}{2} + 1)$. Hence the $\mathbf{S}_k$ operators correspond to spin $\frac{1}{2}$, showing that generalized bivector Dirac fields represent spin $\frac{1}{2}$ fields.

Consider a general boost with rapidity φ along $\mathbf{m}$ (with $\mathbf{m}^2 = 1$). The infinitesimal generator $\mathbf{K_m}$ associated with this transformation is derived similarly to $\mathbf{J_n}$ earlier:

$$\begin{aligned}\psi(\mathbf{x}) &\mapsto \mathrm{e}^{\frac{1}{2}\mathbf{m}\varphi}\psi(\mathrm{e}^{\frac{1}{2}\mathbf{m}\varphi}\mathbf{x}\mathrm{e}^{-\frac{1}{2}\mathbf{m}\varphi}) \\ &= \left(1 + \varphi\left(\tfrac{1}{2}\mathbf{m} + (\mathbf{m} \wedge \mathbf{x}) \cdot \bar{\mathbf{d}}\right) + \mathrm{O}(\varphi^2)\right)\psi(\mathbf{x}).\end{aligned} \tag{60}$$

Thus the infinitesimal generator is $\mathbf{K_m} = \mathrm{i}((\mathbf{x} \wedge \mathbf{m}) \cdot \bar{\mathbf{d}} - \frac{1}{2}\mathbf{m})$ with $\psi \mapsto \mathrm{e}^{\mathrm{i}\mathbf{K_m}\varphi}\psi(\mathbf{x})$. As before define $\mathbf{K}_r = \mathbf{K}_{\mathbf{e}_{0r}}$. Then $\mathbf{K}_r = -\mathrm{i}(\eta(x^0\partial^r + x^r\partial^0) + \frac{1}{2}\mathbf{e}_{0r})$. The total angular momentum operators and boost operators generate a Lie algebra which satisfies the usual relations

$$\mathbf{J}_i \wedge \mathbf{J}_j = \tfrac{1}{2}\mathrm{i}\epsilon_{ij}{}^k\mathbf{J}_k \tag{61}$$

$$\mathbf{K}_i \wedge \mathbf{K}_j = \tfrac{1}{2}\mathrm{i}\epsilon_{ij}{}^k\mathbf{J}_k \tag{62}$$

$$\mathbf{K}_i \wedge \mathbf{J}_j = \tfrac{1}{2}\mathrm{i}\epsilon_{ij}{}^k\mathbf{K}_k. \tag{63}$$

17. Chirality and the Dirac equation

The parity transformation gives rise to the notion of chirality. Denote the left chiral Dirac field by ψ_L and require it to satisfy the Dirac equation $\mathrm{i}\mathbf{d}\psi_L = m\psi_L\mathbf{e}_0$. The action of parity takes the left chiral field to its right chiral counterpart denoted by ψ_R. Thus $\psi_R = -\eta\mathbf{e}_0\psi_L\mathbf{e}_0$. The right chiral field satisfies

$$\mathrm{i}\bar{\mathbf{d}}^\dagger\psi_R = -\eta m\psi_R\mathbf{e}_0. \tag{64}$$

The infinitesimal generators for the restricted Lorentz group corresponding to intrinsic spin are $\mathbf{J}_k = -\mathrm{i}\frac{1}{2}\mathbf{e}_{123}\mathbf{e}_k$ and $\mathbf{K}_r = \frac{1}{2}\mathbf{e}_{0r}$. These Clifford operators satisfy (61)–(63). These operators also generate the Lie group $SU(2) \otimes SU(2)$ with generators $\mathbf{J}_k + \mathrm{i}\mathbf{K}_k$ and $\mathbf{J}_k - \mathrm{i}\mathbf{K}_k$. Moreover, the chiral fields transform according to

$$\psi_L \mapsto \mathrm{e}^{\mathrm{i}(\mathbf{J}_k n^k\theta + \mathrm{i}\mathbf{K}_r m^r\varphi)}\psi_L \tag{65}$$

$$\psi_R \mapsto \mathrm{e}^{\mathrm{i}(\mathbf{J}_k n^k\theta - \mathrm{i}\mathbf{K}_r m^r\varphi)}\psi_R. \tag{66}$$

Consider a vector $\binom{\psi}{\varphi}$ where ψ is a left chiral Dirac field and φ is a right chiral Dirac field. Thus one can construct a grand Dirac equation invariant under the Clifford group $Pin(3)$ (or Lie group $U(2)$).

$$\begin{pmatrix} \mathrm{i}\mathbf{d} & 0 \\ 0 & \mathrm{i}\bar{\mathbf{d}}^\dagger \end{pmatrix}\begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} \psi \\ \varphi \end{pmatrix}\begin{pmatrix} m\mathbf{e}_0 & 0 \\ 0 & -\eta m\mathbf{e}_0 \end{pmatrix}. \tag{67}$$

The parity transformation is given by

$$\begin{pmatrix} \psi \\ \varphi \end{pmatrix} \mapsto -\eta\mathbf{e}_0\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\mathbf{e}_0. \tag{68}$$

18. Interpretation of the plane-wave solution

Recall the general plane-wave solution for a Dirac generalized bivector field propagating along the x-axis. Let $\lambda = \omega + \eta m$ then from (31) we have $\psi = \mathbf{A}(\lambda, k)\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}$ where

$$\mathbf{A}(\lambda, k) = a_1(-\eta\lambda + k\mathbf{e}_{01}) - a_2(k\mathbf{e}_{0123} - \eta\lambda\mathbf{e}_{23}) + a_3(k\mathbf{e}_{02} - \lambda\mathbf{e}_{12}) + a_4(k\mathbf{e}_{03} + \lambda\mathbf{e}_{31}) \tag{69}$$

and the energy is given by $\omega = \pm\sqrt{k^2 + m^2}$. This gives the familiar prediction of particles ($\omega > 0$) and anti-particles ($\omega < 0$). Applying a linear combination of the spin operators S_2 and S_3 to $\mathbf{A}(\lambda, k)$ one can show that no non-trivial choice of a_1, a_2, a_3, a_4 gives rise to an eigenstate.

However, the situation is different for spin along the direction of propagation. Applying the spin operator $\mathbf{S}_1$ to $\mathbf{A}(\lambda, k)$ one finds

$$\begin{aligned}\mathbf{S}_1\mathbf{A}(\lambda, k) &= -\tfrac{1}{2}\eta\mathrm{i}\mathbf{e}_{23}\mathbf{A}(\lambda, k)\\ &= \tfrac{1}{2}\eta\mathrm{i}(a_1(\eta\lambda\mathbf{e}_{23} - k\mathbf{e}_{0123}) + a_2(-k\mathbf{e}_{01} + \eta\lambda) + a_3(k\eta\mathbf{e}_{03} + \lambda\eta\mathbf{e}_{31})\\ &\quad + a_4(-\eta k\mathbf{e}_{02} + \lambda\eta\mathbf{e}_{12})).\end{aligned} \tag{70}$$

Hence $\mathbf{S}_1\psi = \frac{1}{2}\psi$ if and only if $\mathrm{i}\eta a_1 = a_2$, $-\mathrm{i}\eta a_2 = a_1$, $\mathrm{i}a_3 = a_4$ and $-\mathrm{i}a_4 = a_3$. Introducing arbitrary constants a and b one obtains

$$\begin{aligned}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ \mathrm{i}\eta \end{pmatrix} a\\ \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} &= \begin{pmatrix} -\eta \\ -\mathrm{i}\eta \end{pmatrix} b.\end{aligned} \tag{71}$$

Substituting for a_1, a_2, a_3 and a_4 one obtains

$$\mathbf{A}(\lambda, k) = a\mathbf{A}_+(\lambda, k) + b\mathbf{A}_-(\lambda, k) \tag{72}$$

where

$$\mathbf{A}_+(\lambda, k) = (-\eta\lambda + k\mathbf{e}_{01})(1 - \mathrm{i}\eta\mathbf{e}_{23}) \tag{73}$$

$$\mathbf{A}_-(\lambda, k) = (-\eta\lambda + k\mathbf{e}_{01})(\mathbf{e}_{12} + \mathrm{i}\mathbf{e}_{31}). \tag{74}$$

Since $\mathbf{S}_1\mathbf{A}(\lambda, k) = \frac{1}{2}\mathbf{A}(\lambda, k)$ then $\mathbf{S}_1\mathbf{A}^*(\lambda, k) = -\frac{1}{2}\mathbf{A}^*(\lambda, k)$ and interpret $\mathbf{A}(\lambda, k)\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}$ as spin up and $\mathbf{A}^*(\lambda, k)\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}$ as spin down. This spin is along the x-axis and can be thought of as giving rise to left- and right-handed helicities. Each particle spin state is doubly degenerate.

Let P be the parity operator defined earlier in (56). Then

$$P\mathbf{A}_\pm(\lambda, k) = \pm\mathbf{A}_\pm(\lambda, -k). \tag{75}$$

Define the following four chiral fields:

$$\begin{aligned}\psi_{+L} &= A(\lambda, k)_+\mathrm{e}^{\mathrm{i}(\omega t - kx)} & \psi_{+R} &= A(\lambda, -k)_+\mathrm{e}^{\mathrm{i}(\omega t - kx)}\\ \psi_{-L} &= A(\lambda, k)_-\mathrm{e}^{\mathrm{i}(\omega t - kx)} & \psi_{-R} &= A(\lambda, -k)_-\mathrm{e}^{\mathrm{i}(\omega t - kx)}.\end{aligned} \tag{76}$$

The vectors $\binom{\psi_{+L}}{\psi_{+R}}$ and $\binom{\psi_{-L}}{-\psi_{-R}}$ are +1-parity eigenstates of the grand Dirac equation. The vectors $\binom{\psi_{+L}}{-\psi_{+R}}$ and $\binom{\psi_{-L}}{\psi_{-R}}$ are -1-parity eigenstates of the grand Dirac equation. However, the degeneracy remains corresponding to the $U(1) \otimes SU(2)$ global gauge symmetry of the Dirac equation.

19. Conclusion

This paper is a complete and elegant account of the Dirac equation in spacetime algebra. It does not rely on spinors but instead expresses the Dirac field as a generalized bivector field. The Dirac equation is minimally coupled to the electromagnetic potential. Maxwell's electromagnetism is reviewed here for this purpose. The transformation properties of the Dirac equation are investigated and the *CPT* theorem is shown to follow naturally. The infinitesimal generators corresponding to boosts and rotation are derived and found to generate a closed Lie algebra. Parity is used to introduce the notion of chirality which leads to a grand Dirac equation invariant under $Pin(3)$. The plane-wave solutions of the Dirac equation are investigated and found to give eight independent solutions corresponding to particle/anti-particle eigenstates of time reversal, spin eigenstates of $\pm\frac{1}{2}$ along the direction of propagation and two degrees of transverse polarization leave each with a degeneracy of two. This freedom is a result of the internal $U(1) \otimes SU(2)$ symmetry of the Dirac equation. The full implications of this will be given in the next paper of the series.

Acknowledgments

The author acknowledges the support of the Marsden Fund: contract number UOC704. The author would like to thank Hamish Rose for reading various drafts of paper, valuable discussions and his support.

References

[1] Aharonov Y and Bohm D 1959 *Significance of Electromagnetic Potentials in Quantum Theory Phys. Rev.* **115** 484
[2] Clifford W K 1878 *Am. J. Math.* **1** 350
[3] Dirac P A M 1931 Quantised singularities in the electromagnetic field *Proc. R. Soc.* A **113** 60–72
[4] Hestenes D 1966 *Space–Time Algebra* (New York: Gordon and Breach)
[5] Hestenes D 1985 *New Foundations for Classical Mechanics* (Dordrecht: Reidel)
[6] Hestenes D 1996 *Real Dirac Theory* ed J Keller and Z Oziewicz *The Theory of the Electron* (Mexico: Faculted de Estudios Superiores) pp 1–50
[7] Jancewicz B 1988 *Multivectors and Clifford Algebra in Electrodynamics* (Singapore: World Scientific)
[8] Lounesto P 1997 *Clifford Algebras and Spinors (London Mathematical Society Lecture Note Series 239)* (Cambridge: Cambridge University Press)
[9] Ryder L H 1986 *Quantum Field Theory* (Cambridge: Cambridge University Press)
[10] Schubring G (ed) 1996 *Grassmann's Vision* (Dordrecht: Kluwer) pp 243–54